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Error calculus and regularity of Poisson functionals: the lent particle method.

Nicolas BOULEAU*

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Abstract

We propose a new method to apply the Lipschitz functional calculus of local Dirichlet forms to Poisson random measures.

Résumé

Calcul d'erreur et régularité des fonctionnelles de Poisson : la méthode de la particule prêtée. Nous proposons une nouvelle méthode pour appliquer le calcul fonctionnel lipschitzien des formes de Dirichlet locales aux mesures aléatoires de Poisson.

1 Notation and basic formulae.

Let us consider a local Dirichlet structure with carré du champ $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ where (X, \mathcal{X}, ν) is a σ -finite measured space called *bottom-space*. Singletons are in \mathcal{X} and ν is diffuse, \mathbf{d} is the domain of the Dirichlet form $\epsilon[u] = 1/2 \int \gamma[u] d\nu$. We denote $(a, \mathcal{D}(a))$ the generator in $L^2(\nu)$ (cf. [3]).

A random Poisson measure associated to (X, \mathcal{X}, ν) is denoted N . Ω is the configuration space of countable sums of Dirac masses on X and \mathcal{A} is the σ -field generated by N , of law \mathbb{P} on Ω . The space $(\Omega, \mathcal{A}, \mathbb{P})$ is called *the up-space*. We write $N(f)$ for $\int f dN$. If $p \in [1, \infty[$ the set $\{e^{iN(f)} : f \text{ real}, f \in L^1 \cap L^2(\nu)\}$ is total in $L^p_{\mathbb{C}}(\Omega, \mathcal{A}, \mathbb{P})$. We put $\tilde{N} = N - \nu$. The relation $\mathbb{E}(\tilde{N}f)^2 = \int f^2 d\nu$ extends and gives sense to $\tilde{N}(f)$, $f \in L^2(\nu)$. The Laplace functional and the differential calculus with γ yield

$$(1) \quad \forall f \in \mathbf{d}, \forall h \in \mathcal{D}(a) \quad \mathbb{E}[e^{i\tilde{N}(f)}(\tilde{N}(a[h]) + \frac{i}{2}N(\gamma[f, h]))] = 0.$$

2 Product, particle by particle, of a Poisson random measure by a probability measure.

Given a probability space (R, \mathcal{R}, ρ) , let us consider a Poisson random measure $N \odot \rho$ on $(X \times R, \mathcal{X} \times \mathcal{R})$ with intensity $\nu \times \rho$ such that for $f \in L^1(\nu)$ and $g \in L^1(\rho)$ if

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$N(f) = \sum f(x_n)$ then $(N \odot \rho)(fg) = \sum f(x_n)g(r_n)$ where the r_n 's are i.i.d. independent of N with law ρ . Calling $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{P}})$ the product of all the factors (R, \mathcal{R}, ρ) involved in the construction of $N \odot \rho$, we obtain the following properties : For an $\mathcal{A} \times \mathcal{X} \times \mathcal{R}$ -measurable and positive function F , $\hat{\mathbb{E}} \int F(\omega, x, r) N \odot \rho(dx dr) = \int F d\rho dN$ \mathbb{P} -a.s.

Let us denote by \mathbb{P}_N the measure $\mathbb{P}(d\omega)N_\omega(dx)$ on $(\Omega \times X, \mathcal{A} \times \mathcal{X})$. We have

Lemma 2.1 *Let F be $\mathcal{A} \times \mathcal{X} \times \mathcal{R}$ -measurable, $F \in \mathcal{L}^2(\mathbb{P}_N \times \rho)$ and such that $\int F(\omega, x, r) \rho(dr) = 0$ \mathbb{P}_N -a.s., then $\int F d(N \odot \rho)$ is well defined, belongs to $L^2(\mathbb{P} \times \hat{\mathbb{P}})$ and*

$$(2) \quad \hat{\mathbb{E}} \left(\int F d(N \odot \rho) \right)^2 = \int F^2 dN d\rho \quad \mathbb{P}\text{-a.s.}$$

The argument consists in considering F_n satisfying

$$\mathbb{E} \int F_n^2 d\nu d\rho < +\infty \text{ and } \mathbb{E} \int (\int |F_n| d\nu)^2 d\rho < +\infty \text{ and then using the relation}$$

$$\hat{\mathbb{E}} \left(\int F_n d(N \odot \rho) \right)^2 = \left(\int F_n d\rho dN \right)^2 - \int \left(\int F_n d\rho \right)^2 dN + \int F_n^2 d\rho dN \quad \mathbb{P}\text{-a.s.}$$

3 Construction by Friedrichs' method and expression of the gradient.

a) We suppose the space by \mathbf{d} of the bottom structure is separable, then a gradient exists (cf. [3] Chap. V, p.225 *et seq.*). We denote it \flat and choose it with values in the space $L^2(R, \mathcal{R}, \rho)$. Thus, for $u \in \mathbf{d}$ we have $u^\flat \in L^2(\nu \times \rho)$, $\gamma[u] = \int (u^\flat)^2 d\rho$ and \flat satisfies the chain rule. We suppose in addition, what is always possible, that \flat takes its values in the subspace orthogonal to the constant 1, i.e.

$$(3) \quad \forall u \in \mathbf{d} \quad \int u^\flat d\rho = 0 \quad \nu\text{-a.s.}$$

This hypothesis is important here as in many applications (cf. [2] Chap V §4.6). We suppose also, but this is not essential (cf. [3] p44) $1 \in \mathbf{d}_{loc}$ $\gamma[1] = 0$ so that $1^\flat = 0$.

b) We define a pre-domain \mathcal{D}_0 dense in $L^2_{\mathbb{C}}(\mathbb{P})$ by

$$\mathcal{D}_0 = \left\{ \sum_{p=1}^m \lambda_p e^{i\tilde{N}(f_p)}; m \in \mathbb{N}^*, \lambda_p \in \mathbb{C}, f_p \in \mathcal{D}(a) \cap L^1(\nu) \right\}.$$

c) We introduce the creation operator inspired from quantum mechanics (see [7], [8], [9], [1], [5],[6] and [10] among others) defined as follows

$$(4) \quad \varepsilon_x^+(\omega) \text{ equals } \omega \text{ if } x \in \text{supp}(\omega), \text{ and equals } \omega + \varepsilon_x \text{ if } x \notin \text{supp}(\omega)$$

so that

$$(5) \quad \varepsilon_x^+(\omega) = \omega \quad N_\omega\text{-a.e. } x \quad \text{and} \quad \varepsilon_x^+(\omega) = \omega + \varepsilon_x \quad \nu\text{-a.e. } x$$

This map is measurable and the Laplace functional shows that for an $\mathcal{A} \times \mathcal{X}$ -measurable $H \geq 0$,

$$(6) \quad \mathbb{E} \int \varepsilon^+ H d\nu = \mathbb{E} \int H dN.$$

Let us remark also that by (5), for $F \in \mathcal{L}^2(\mathbb{P}_N \times \rho)$

$$(7) \quad \int \varepsilon^+ F d(N \odot \rho) = \int F d(N \odot \rho) \quad \mathbb{P} \times \hat{\mathbb{P}}\text{-a.s.}$$

d) We defined a gradient \sharp for the up-structure on \mathcal{D}_0 by putting for $F \in \mathcal{D}_0$

$$(8) \quad F^\sharp = \int (\varepsilon^+ F)^\flat d(N \odot \rho)$$

this definition being justified by the fact that for \mathbb{P} -a.e. ω the map $y \mapsto F(\varepsilon_y^+(\omega)) - F(\omega)$ is in \mathbf{d} , $\varepsilon^+ F$ belongs to $L^\infty(\mathbb{P}) \otimes \mathbf{d}$ algebraic tensor product, and $(\varepsilon^+ F - F)^\flat = (\varepsilon^+ F)^\flat \in L^2(\mathbb{P}_N \times \rho)$.

For $F, G \in \mathcal{D}_0$ of the form

$$F = \sum_p \lambda_p e^{i\tilde{N}(f_p)} = \Phi(\tilde{N}(f_1), \dots, \tilde{N}(f_m)) \quad G = \sum_q \mu_q e^{i\tilde{N}(g_q)} = \Psi(\tilde{N}(g_1), \dots, \tilde{N}(g_n))$$

we compute using (2), (3) and (7) (in the spirit of prop. 1 of [9] or lemma 1.2 of [6])

$$(9) \quad \mathbb{E}[F^\sharp \overline{G^\sharp}] = \sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p) - i\tilde{N}(g_q)} \gamma[f_p, g_q]$$

and we have

Proposition 3.1 *If we put $A_0[F] = \sum_p \lambda_p e^{i\tilde{N}(f_p)} (i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p]))$ it comes*

$$(10) \quad \mathbb{E}[A_0[F] \overline{G}] = -\frac{1}{2} \mathbb{E} \sum_{p,q} \Phi'_p \overline{\Psi'_q} N(\gamma[f_p, g_q]).$$

In order to show that $A_0[F]$ does not depend on the form of F , by (10) it is enough to show that the expression $\sum_{p,q} \Phi'_p \overline{\Psi'_q} N(\gamma[f_p, g_q])$ depends only on F and G . But this comes from (9) since F^\sharp and G^\sharp depend only on F and G .

By this proposition, A_0 is symmetric on \mathcal{D}_0 , negative, and the argument of Friedrichs applies (cf [3] p4), A_0 extends uniquely to a selfadjoint operator $(A, \mathcal{D}(A))$ which defines a closed positive (hermitian) quadratic form $\mathcal{E}[F] = -\mathbb{E}[A[F] \overline{F}]$. By (10) contractions operate and (cf. [3]) \mathcal{E} is a Dirichlet form which is local with carré du champ denoted Γ and the up-structure obtained $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies

$$(11) \quad \forall f \in \mathbf{d}, \quad \tilde{N}(f) \in \mathbb{D} \quad \text{and} \quad \Gamma[\tilde{N}(f)] = N(\gamma[f])$$

The operator \sharp extends to a gradient for Γ as a closed operator from $L^2(\mathbb{P})$ into $L^2(\mathbb{P} \times \hat{\mathbb{P}})$ with domain \mathbb{D} which satisfies the chain rule and may be computed on functionals $\Phi(\tilde{N}(f_1), \dots, \tilde{N}(f_m))$, Φ Lipschitz and \mathcal{C}^1 and their limits in \mathbb{D} (as done in [4]).

Formula (8) for \sharp can be extended from \mathcal{D}_0 to \mathbb{D} . Let us introduce the space $\underline{\mathbb{D}}$ closure of $\mathcal{D}_0 \otimes \mathbf{d}$ for the norm

$$\|H\|_{\underline{\mathbb{D}}} = (\mathbb{E} \int \gamma[H(\omega, \cdot)](x) N(dx))^{1/2} + \mathbb{E} \int |H(\omega, x)| \xi(x) N(dx)$$

where $\xi > 0$ is a fixed function such that $N(\xi) \in L^2(\mathbb{P})$.

Theorem 3.1 *The formula $F^\sharp = \int (\varepsilon^+ F)^\flat d(N \odot \rho)$ decomposes as follows*

$$F \in \mathbb{D} \xrightarrow{\varepsilon^+} \varepsilon^+ F \in \underline{\mathbb{D}} \xrightarrow{\flat} (\varepsilon^+ F)^\flat \in L_0^2(\mathbb{P}_N \times \rho) \xrightarrow{d(N \odot \rho)} F^\sharp \in L^2(\mathbb{P} \times \hat{\mathbb{P}})$$

where each operator is continuous on the range of the preceding one, $L_0^2(\mathbb{P}_N \times \rho)$ denoting the closed subspace of $L^2(\mathbb{P}_N \times \rho)$ of ρ -centered elements, and we have

$$(12) \quad \Gamma[F] = \hat{\mathbb{E}}|F^\sharp|^2 = \int \gamma[\varepsilon^+ F] dN.$$

4 The lent particle method.

Let us consider, for instance, a real process Y_t with independent increments and Lévy measure σ integrating x^2 , Y_t being supposed centered without Gaussian part. We assume that σ has an l.s.c. density so that a local Dirichlet structure may be constructed on $\mathbb{R} \setminus \{0\}$ with carré du champ $\gamma[f] = x^2 f'^2(x)$. If N is the random Poisson measure with intensity $dt \times \sigma$ we have $\int_0^t h(s) dY_s = \int 1_{[0,t]}(s) h(s) x \tilde{N}(ds dx)$ and the choice done for γ gives $\Gamma[\int_0^t h(s) dY_s] = \int_0^t h^2(s) d[Y, Y]_s$ for $h \in L_{loc}^2(dt)$. In order to study the regularity of the random variable $V = \int_0^t \varphi(Y_{s-}) dY_s$ where φ is Lipschitz and \mathcal{C}^1 , we have two ways:

a) We may represent the gradient \sharp as $Y_t^\sharp = B_{[Y, Y]_t}$ where B is a standard auxiliary independent Brownian motion. Then by the chain rule $V^\sharp = \int_0^t \varphi'(Y_{s-})(Y_{s-})^\sharp dY_s + \int_0^t \varphi(Y_{s-}) dB_{[Y]_s}$ now, using $(Y_{s-})^\sharp = (Y_s^\sharp)_-$, a classical but rather tedious stochastic computation yields

$$(13) \quad \Gamma[V] = \hat{\mathbb{E}}[V^{\sharp 2}] = \sum_{\alpha \leq t} \Delta Y_\alpha^2 \left(\int_{\alpha}^t \varphi'(Y_{s-}) dY_s + \varphi(Y_{\alpha-}) \right)^2.$$

Since V has real values the *energy image density property* holds, and V has a density as soon as $\Gamma[V]$ is strictly positive a.s. what may be discussed using the relation (13).

b) Another more direct way consists in applying the theorem. For this we define \flat by choosing η such that $\int_0^1 \eta(r) dr = 0$ and $\int_0^1 \eta^2(r) dr = 1$ and putting $f^\flat = x f'(x) \eta(r)$.

1°. First step. We add a particle (α, x) i.e. a jump to Y at time α with size x what gives

$$\varepsilon^+ V - V = \varphi(Y_{\alpha-})x + \int_{\alpha}^t (\varphi(Y_{s-} + x) - \varphi(Y_{s-})) dY_s$$

2°. $V^\flat = 0$ since V does not depend on x , and

$$(\varepsilon^+ V)^\flat = (\varphi(Y_{\alpha-})x + \int_{\alpha}^t \varphi'(Y_{s-} + x) x dY_s) \eta(r) \quad \text{because } x^\flat = x \eta(r).$$

3°. We compute $\gamma[\varepsilon^+ V] = \int (\varepsilon^+ V)^\flat{}^2 dr = (\varphi(Y_{\alpha-})x + \int_{\alpha}^t \varphi'(Y_{s-} + x) x dY_s)^2$

4°. We take back the particle we gave, because in order to compute $\int \gamma[\varepsilon^+ V] dN$ the integral in N confuses $\varepsilon^+ \omega$ and ω .

That gives $\int \gamma[\varepsilon^+ V] dN = \int (\varphi(Y_{\alpha-}) + \int_{\alpha}^t \varphi'(Y_{s-}) dY_s)^2 x^2 N(d\alpha dx)$ and (13).

We remark that both operators $F \mapsto \varepsilon^+ F$, $F \mapsto (\varepsilon^+ F)^\flat$ are non-local, but instead $F \mapsto \int (\varepsilon^+ F)^\flat d(N \odot \rho)$ and $F \mapsto \int \gamma[\varepsilon^+ F] dN$ are local : taking back the lent particle gives the locality.

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